Minimal Discriminants for Elliptic Curves with Non-Trivial Isogeny

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Outline

Elliptic Curves

Definition

*An **Elliptic Curve** over $\mathbb{Q}$ is the set of complex numbers (x, y) that satisfy the equation*

$$y^2 = x^3 + Ax + B$$

together with a point "at infinity" denoted $\mathcal{O}$, where $A, B \in \mathbb{Q}$ satisfy $4A^3 + 27B^2 \neq 0$.

Why are Elliptic Curves Important?

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- The "applications" answer
  - Cryptography
- The "mathematics" answer
  - Bridge between algebra and geometry

Elliptic Curve Theorems

Theorem (Mordell-Weil, 1922)

The set of rational points $E(\mathbb{Q})$ has the structure of a finitely generated abelian group with identity element $\mathcal{O}$.

Theorem (Mazur, 1977)

Let E be an elliptic curve over $\mathbb{Q}$. Then the torsion subgroup, the subgroup of points of finite order, is isomorphic to one of the following possibilities:

$$E(\mathbb{Q})_{tors} \cong \begin{cases} C_N, & N = 1, 2, \ldots, 10, 12 \\ C_2 \times C_N, & N = 1, 2, 3, 4. \end{cases}$$

Weierstrass Form of an Elliptic Curve

The **Weierstrass form** of an elliptic curve over $\mathbb{Q}$ is given by

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

where each $a_j \in \mathbb{Q}$. We say E is given by an **integral Weierstrass model** if each $a_j \in \mathbb{Z}$.

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Define the quantities associated to E by

$$\begin{aligned} c_4 &= a_1^4 + 8a_1^2a_2 - 24a_1a_3 - 48a_4 \\ c_6 &= -(a_1^2 + 4a_2)^3 + 36(a_1^2 + 4a_2)(2a_4 + a_1a_3) - 216(a_3^2 + 4a_6) \\ \Delta &= \frac{c_4^3 - c_6^2}{1728}, \qquad j(E) = \frac{c_4^3}{\Delta}. \end{aligned}$$

We call Δ the **discriminant** and $j(E)$ the j-**invariant** of E.

Isomorphisms of Elliptic Curves

An elliptic curve E' is $\mathbb{Q}$-isomorphic to E if E' arises from E via an **admissible change of variables**

$$x \longmapsto u^2x + r \qquad y \longmapsto u^3y + u^2sx + w,$$

where $u, r, s, w \in \mathbb{Q}$ and $u \neq 0$.

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Let c_4', c_6', Δ', and j' be the quantities associated to E'. Then,

$$c_4' = u^{-4}c_4, \quad c_6' = u^{-6}c_6, \quad \Delta' = u^{-12}\Delta, \quad j' = j$$

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Composition of isomorphisms affects u multiplicatively:

$$E_1 \xrightarrow{u_1} E_2 \xrightarrow{u_2} E_3 \implies \Delta_3 = u_2^{-12}\Delta_2 = u_2^{-12}u_1^{-12}\Delta_1$$

Examples

Suppose we have elliptic curves

$$E : y^2 + 81xy + 24786y = x^3 + 324x^2$$
$$E' : y^2 + xy = x^3 - 43x + 105.$$

They are isomorphic via the change of variables

$$x \longmapsto 9^2x - 648 \quad y \longmapsto 9^3y - 9^2 \cdot 36x + 13851.$$

That is, $(u, r, s, w) = (9, -648, -36, 13851)$.

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One can show that E has discriminant 652977088344072 and E' has discriminant 2312.
Note that

$$652977088344072 = 2^3 \cdot 3^{24} \cdot 17^2 \quad \text{and} \quad 2312 = 2^3 \cdot 17^2.$$

GeoGebra example!

Minimal Discriminants

We say E defined by

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

is a **global minimal model** if each $a_j \in \mathbb{Z}$ and Δ is minimal over all $\mathbb{Q}$-isomorphic curves:

$$\Delta_E = \min\{|\Delta_{E'}| \in \mathbb{Z} : \Delta_{E'} \text{ is the discriminant of } E' \in [E]_{\mathbb{Q}}\}$$

The discriminant associated with a global minimal model is called the **minimal discriminant**.

Additive Reduction

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- We say E is **semistable at a prime** p if it does not have additive reduction at a prime p.
- We say E is **semistable** if E is semistable at every prime.

Additive Reduction

Example

Suppose we have the elliptic curves

$$E : y^2 + 81xy + 24786y = x^3 + 324x^2$$

$$E' : y^2 + xy = x^3 - 43x + 105.$$

A global minimal model of E is given by E'.
We saw that E' has minimal discriminant $2312 = 2^3 \cdot 17^2$.
We have that $\Delta_{E'}^{\min} = 2^3 \cdot 17^2$ and $c_4 = 5 \cdot 7 \cdot 59$, so we have that E' is semistable.

Computing Minimal Discriminants

- Tate's algorithm (1975)
- Laska's algorithm (1982)
- Kraus-Laska-Connell algorithm (1991)

Frey Curve

The **Frey Curve**, named for Gerhard Frey, is defined by

$$F(a,b) : y^2 = x(x+a)(x-b),$$

where a and b are coprime positive integers with a even. Its discriminant is $\Delta = (4ab(a+b))^2$.

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Theorem (Hellegouarch, 1975)

The minimal discriminant of $F(a,b)$ is $\Delta^{min} = u^{-12}\Delta$, where

$$u = \begin{cases} 2 & \textit{if } a \equiv 0 \pmod{16} \textit{ and } b \equiv 3 \pmod 4 \\ 1 & \textit{otherwise.} \end{cases}$$

Fermat's Last Theorem

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Theorem (Wiles and Taylor, 1995)

Fermat's equation

$$x^n + y^n = z^n$$

has no integer solutions for $n \geq 3$ *such that* $xyz \neq 0$

- Consider the corresponding Frey Curve $F(a^n, b^n) : y^2 = x(x + a^n)(x - b^n)$ taking the values to make $F(a^n, b^n)$ to be semi-stable.
- If Fermat's last theorem were not true, this curve would not be modular

Minimal Discriminant for Frey Curve

- $\gcd(a, b, c) = 1$ which means exactly one of a^n, b^n or c^n must be even,so we can relabel and call the even term a^n.
- Similarly, we can rearrange terms so $b^n \equiv 3 \mod 4$.
- For $F(a^p, b^p) : y^2 = x(x + a^p)(x - b^p)$ where $p \geq 5$, the minimal discriminant is:$\left(\frac{a^p b^p c^p}{16}\right)^2$.
- $a^p \equiv 0 \mod 16$ and $b^p \equiv 3 \mod 4$.

Extension of Hellegouarch

- The Frey curve comes equipped with an easily computable minimal discriminant.
- Barrios extended Hellegourch's result to all elliptic curves with a non-trivial torsion subgroup.
- We focused on extending this result to all elliptic curves that have a non-trivial isogeny.

Kraus' Theorem

Theorem (1989)

Let $\alpha, \beta, \gamma \in \mathbb{Z}$ *with* $\gamma \neq 0$ *be such that* $\alpha^3 - \beta^2 = 1728\gamma$*. There exists an integral Weierstrass model with* $c_4 = \alpha$ *and* $c_6 = \beta$ *if and only if*

1. $v_3(\beta) \neq 2$*, and*
2. - $\beta \equiv -1 \pmod 4$ *if* β *is odd*
   - $v_2(\alpha) \geq 4$ *and* $\beta \equiv 0$ *or* $8 \pmod{32}$ *if* β *is even.*

Kraus' Theorem Example

- We verify that $F : y^2 + 18xy + 189y = x^3$ is an integral model using Kraus' Theorem. Note that for $F_{9,2}$, we have
- $c_4 = 9(36a^2 - 6ab + b^2)(6a + b)b$ and $c_6 = -27(324a^4 - 108a^3b + 54a^2b^2 + 6ab^3 + b^4)(18a^2 + 6ab - b^2)$.
- Plugging in $a = 1$ and $b = 6$ yields $c_4 = 23328$ and $c_6 = -2047032$. This means that $v_3(c_6) = 9$, $v_2(c_4) = 5$, and $c_6 \equiv 8 \pmod{32}$. Kraus tells us that an integral model with these invariants exists!

Isogenies

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- When this occurs, we say that E and E' are **isogenous**.
- We say an isogeny has degree N if $|\ker \pi| = N$.
- In particular, a **cyclic** isogeny of degree N has $\ker \pi \cong C_N$. An isogeny of degree N is also called an N-isogeny.

Isogenies

- If $E : y^2 = x^3 + Ax + B$ and $E' : y^2 = x^3 + A'x + B'$ then an isogeny $\phi : E \to E'$ can be written as

$$\phi(x, y) = \left(f(x), c\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)$$

for some $f(x) \in \mathbb{Q}(x)$ with $c \in \mathbb{Q}$ and $c \neq 0$.

Example of Isogeny

- Taking 2 curves in the 8-isogeny,
  $a4 : y^2 = x^3 - 23003136x + 31708938240$ and
  $a2 : y^2 = x^3 - 21344256x + 37951635456$
- $f(x) = \frac{x^2 - 2688\,x + 331776}{x - 2688}$ and $c = 1$

ISogeny.png

Figure: 24.a Isogeny Class

Example of Isogeny

24.a1.png

Figure: 24.a1

24.a2.png

Figure: 24.a2

Modular Curves

We say that $(E_1, E_1', \pi_1) \sim (E_2, E_2', \pi_2)$ if and only if there exist isomorphisms $\phi : E_1 \to E_2$ and $\phi' : E_1' \to E_2'$ such that

$$\begin{tikzcd} E_1 \arrow[r, "\pi_1"] \arrow[d, "\phi"] & E_1' \arrow[d, "\phi'"] \\ E_2 \arrow[r, "\pi_2"] & E_2' \end{tikzcd}$$

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$$\begin{array}{ccc} E_1 & \xrightarrow{\pi_1} & E_1' \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ E_2 & \xrightarrow{\pi_2} & E_2' \end{array}$$

Definition (Modular Curves)

The modular curve $X_0(N)$ parameterizes isomorphism classes of triples (E, E', π), where $\pi : E \to E'$ is a cyclic N-isogeny.

Here we consider $N = 1, 2, \ldots, 10, 12, 13, 16, 18, 25$. This is where the genus of $X_0(N)$ is 0.

These parameterizations are made explicit:

Fricke Parameterizations

If two elliptic curves E and E' are isogenous over $\mathbb{Q}$, there exists $t \in \mathbb{Q}$ such that the j-invariants of E and E' are given by $j_{n,1}(t)$ and $j_{n,2}(t)$, respectively:

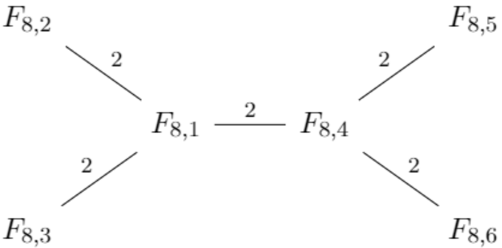
TABLE 1. The Fricke Parameterizations: j-invariants $j_{n,i}$

n	$j_{n,1}(t)$	$j_{n,2}(t)$
6	$\frac{(t+12)^3(t^3+252t^2+3888t+15552)^3}{t^6(t+8)^2(t+9)^3}$	$\frac{(t+6)^3(t^3+18t^2+84t+24)^3}{t(t+8)^3(t+9)^2}$
8	$\frac{(t^4+240t^3+2144t^2+3840t+256)^3}{t(t-4)^8(t+4)^2}$	$\frac{(t^4-16t^2+16)^3}{t^2(t^2-16)}$
9	$\frac{(t+6)^3(t^3+234t^2+756t+2160)^3}{(t-3)^8(t^3-27)}$	$\frac{t^3(t^3-24)^3}{t^3-27}$

Parameterizations exist for the other values of N, but they are omitted.

Fricke Parameterizations

Let $n \geq 2$ be an integer such that $X_0(n)$ has genus 0. As part of our research project, we consider various parameterized families of elliptic curves $F_{n,i}(a, b, d)$ with the property that they parameterize isogenous elliptic curves that admit a degree n isogeny.



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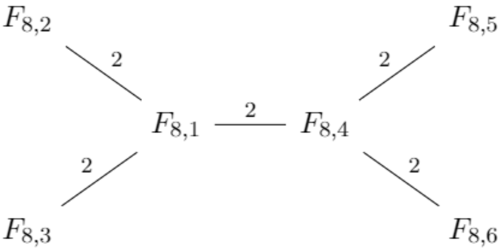

These families are related to the Fricke parameterizations by the following theorem:

Fricke Parameterizations

Theorem (Barrios)

Let $n \geq 2$ be an integer such that $X_0(n)$ has genus 0 and suppose E is a rational elliptic curve such that its isogeny degree is n. Then there are integers a, b, d such that $\gcd(a, b) = 1$ and the following hold:

(1) E is $\mathbb{Q}$-isomorphic to $F_{n,k}(a, b, d)$ for some k,

(2) $j_{n,1}\left(\frac{b}{a}\right) = j(F_{n,l}(a, b, d))$ and $j_{n,2}\left(\frac{b}{a}\right) = j(F_{n,h}(a, b, d))$ for some l and h,

(3) The isogeny class of E is $\left\{[F_{n,i}(a, b, d)]_{\mathbb{Q}}\right\}_i$.

Above, $j_{n,i}(t)$ refers to the Fricke parameterization.

Our Task

We aim to classify the minimal discriminants of elliptic curves with non-trivial isogeny.

So far, we have classified the minimal discriminants of 6-, 8-, and 9-isogenous elliptic curves in terms of arithmetic conditions on the parameters a and b, taking $d = 1$.

Our Task

Lemma

If E is a rational elliptic curve given by an integral Weierstrass model with invariants c_4 and c_6 and discriminant Δ, then there is a unique positive integer u such that

$$c_4' = u^{-4}c_4, \qquad c_6' = u^{-6}c_6, \qquad \textit{and} \qquad \Delta_E^{\textit{min}} = u^{-12}\Delta$$

where $\Delta_E^{\textit{min}}$ is the minimal discriminant of E and c_4' and c_6' are the invariants associated to a global minimal model of E.

Main Theorem

Theorem (B.,E.,F.,M.,S.)

Let $n = 6,\ 8,$ or 9 and consider the elliptic curves $F_{n,i} = F_{n,i}(a, b, 1)$. Let $\Delta_{n,i}$ denote the discriminant of $F_{n,i}$. Then the minimal discriminant of $F_{n,i}$ is $u^{-12}\Delta_{n,i}$ where u is uniquely determined from the p-adic valuations given in the following table:

Results

n	p	Condition on a, b	$(v_p(u_{n,i}))_i$
6	2	$v_2(b) = 0$	$(1, 0, 1, 2)$
		$v_2(b) = 1$	$(2, 0, 1, 2)$
		$v_2(b) = 2$	$(3, 0, 2, 2)$
		$v_2(b) \geq 3$	$(3, 1, 3, 3)$
	3	$v_3(b) = 0$	$(0, 0, 0, 0)$
		$v_3(b) = 1$	$(1, 1, 0, 0)$
		$v_3(b) \geq 2$	$(2, 2, 1, 1)$
8	2	$v_2(b) = 0$	$(1, 2, ?, 0, 0, 1)$
		$v_2(b) = 1$	$(2, ?, ?, 1, 1, 2)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \geq 4$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \leq 3$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 0$	$(3, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$
9	3	$v_3(b) = 0$	$(1, 0, 0)$
		$v_3(b) \geq 1$ and $v_3(a - \frac{b}{3}) = 0$	$(1, 1, 0)$
		$v_3(b) = 1$ and $v_3(a - \frac{b}{3}) = 1$	$(2, 1, 0)$
		$v_3(b) = 1$ and $v_3(a - \frac{b}{3}) > 1$	$(3, 2, 1)$

How to Use the Table

This is the table that displays our results for the 6 curves of the 8-Isogeny:

8	2	$v_2(b)=0$	$(1,2,?,0,0,1)$
		$v_2(b)=1$	$(2,?,?,1,1,2)$
		$v_2(b)=2$ and $v_2(a^2-\frac{b^2}{16})\geq 4$	$(5,?,?,?,3,3)$
		$v_2(b)=2$ and $v_2(a^2-\frac{b^2}{16})\leq 3$	$(4,?,?,?,2,2)$
		$v_2(b)=2$ and $v_2(a+\frac{b}{4})=0$	$(3,?,?,?,2,2)$
		$v_2(b)=2$, $v_2(a+\frac{b}{4})=1$, and $v_2(a-\frac{b}{4})\leq 2$	$(4,?,?,?,2,2)$
		$v_2(b)=2$, $v_2(a+\frac{b}{4})=1$, and $v_2(a-\frac{b}{4})\geq 3$	$(5,?,?,?,3,3)$
		$v_2(b)=2$ and $v_2(a+\frac{b}{4})=2$	$(5,?,?,?,2,2)$
		$v_2(b)=2$ and $v_2(a+\frac{b}{4})\geq 3$	$(5,?,?,?,3,3)$
		$v_2(b)\geq 3$	$(3,?,?,2,3,2)$

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But how do we read it? Lets try some examples.

How to Use the Table: curve 6, $a = 1$, $b = 7$

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Example: Try to find the u value of the 6th curve of the 8-Isogeny when $a = 1$ $b = 7$. $F_{8,6}(1,7) : y^2 = x^3 - 164x^2 + 256x$
$b = 7 = 7 \cdot 1 = 7 \cdot 2^0$. So the $v_2(b) = 0$.

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		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$

How to Use the Table: curve 6, $a = 1$, $b = 7$

Find the condition that is satisfied when $v_2(b) = 0$.

8	2	$v_2(b) = 0$	$(1, 2, ?, 0, 0, 1)$
		$v_2(b) = 1$	$(2, ?, ?, 1, 1, 2)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \geq 4$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \leq 3$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 0$	$(3, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$

How to Use the Table: curve 6, $a = 1$, $b = 7$

Now since we are finding the u value when for the 6th curve of the 8-Isogeny, we look at the 6th column to find our answer.

8	2	$v_2(b) = 0$	$(1, 2, ?, 0, 0, 1)$
		$v_2(b) = 1$	$(2, ?, ?, 1, 1, 2)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \geq 4$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \leq 3$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 0$	$(3, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$

How to Use the Table: curve 6, $a = 1$, $b = 7$

Now since we are finding the u value when for the 6th curve of the 8-Isogeny, we look at the 6th column to find our answer.

8	2	$v_2(b) = 0$	$(1, 2, ?, 0, 0, 1)$
		$v_2(b) = 1$	$(2, ?, ?, 1, 1, 2)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \geq 4$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \leq 3$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 0$	$(3, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$

So for the 6th curve of the 8-Isogeny, $v_2(u) = 1$, so $u = 2$ when $a = 1$ and $b = 7$.

How to Use the Table: Curve 1, $a = 59$, $b = 20$

Now lets try this example: Find the u value of the 1st curve of the 8-isogeny when $a = 59$ and $b = 20$.

$F_{8,1}(59, 20) : y^2 + 160xy - 35389440y = x^3 - 221184x^2$

$b = 20 = 4 \cdot 5 = 2^2 \cdot 5$. So $v_2(b) = 2$.

$a + \frac{b}{4} = 59 + \frac{20}{4} = 64 = 2^6$. So $v_2(a + \frac{b}{4}) = 6$.

8	2	$v_2(b) = 0$	$(1, 2, ?, 0, 0, 1)$
		$v_2(b) = 1$	$(2, ?, ?, 1, 1, 2)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \geq 4$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \leq 3$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 0$	$(3, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$

How to Use the Table: Curve 1, $a = 59$, $b = 20$

Find the condition that is satisfied when $v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 6$.

8	2	$v_2(b) = 0$	$(1, 2, ?, 0, 0, 1)$
		$v_2(b) = 1$	$(2, ?, ?, 1, 1, 2)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \geq 4$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \leq 3$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 0$	$(3, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$

How to Use the Table: Curve 1, $a = 59$, $b = 20$

Now since we are finding the u value when for the 1st curve of the 8-Isogeny, we look at the 1st column to find our answer.

8	2	$v_2(b) = 0$	$(1, 2, ?, 0, 0, 1)$
		$v_2(b) = 1$	$(2, ?, ?, 1, 1, 2)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \geq 4$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \leq 3$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 0$	$(3, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$

How to Use the Table: Curve 1, $a = 59$, $b = 20$

Now since we are finding the u value when for the 1st curve of the 8-Isogeny, we look at the 1st column to find our answer.

8	2	$v_2(b) = 0$	$(1, 2, ?, 0, 0, 1)$
		$v_2(b) = 1$	$(2, ?, ?, 1, 1, 2)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \geq 4$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \leq 3$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 0$	$(3, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$

So for the 1st curve of the 8-Isogeny, $v_2(u) = 5$, so $u = 32$ when $a = 59$ and $b = 20$.

How to Use the Table: Curve 5, $a = 117$, $b = 68$

One more example: Find the u value of the 5th curve of the 8-isogeny when $a = 117$ and $b = 68$.

$F_{8,5}(117, 68) : y^2 = x^3 - 866848x^2 + 21381376x$

$b = 68 = 4 \cdot 17 = 2^2 \cdot 17$. So $v_2(b) = 2$.

$a + \frac{b}{4} = 117 + \frac{68}{4} = 134 = 2 \cdot 67$. So $v_2(a + \frac{b}{4}) = 1$.

$a - \frac{b}{4} = 117 - \frac{68}{4} = 100 = 4 \cdot 25 = 2^2 \cdot 25$. So $v_2(a - \frac{b}{4}) = 2$.

8	2	$v_2(b) = 0$	$(1, 2, ?, 0, 0, 1)$
		$v_2(b) = 1$	$(2, ?, ?, 1, 1, 2)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \geq 4$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \leq 3$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 0$	$(3, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$

How to Use the Table: Curve 5, $a = 117$, $b = 68$

Find the condition that is satisfied when $v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) = 2$.

8	2	$v_2(b) = 0$	$(1, 2, ?, 0, 0, 1)$
		$v_2(b) = 1$	$(2, ?, ?, 1, 1, 2)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \geq 4$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \leq 3$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 0$	$(3, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$

How to Use the Table: Curve 5, $a = 117$, $b = 68$

Now since we are finding the u value when for the 5th curve of the 8-Isogeny, we look at the 5th column to find our answer.

8	2	$v_2(b) = 0$	$(1, 2, ?, 0, 0, 1)$
		$v_2(b) = 1$	$(2, ?, ?, 1, 1, 2)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \geq 4$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \leq 3$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 0$	$(3, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$

How to Use the Table: Curve 5, $a = 117$, $b = 68$

Now since we are finding the u value when for the 5th curve of the 8-Isogeny, we look at the 5th column to find our answer.

8	2	$v_2(b) = 0$	$(1, 2, ?, 0, 0, 1)$
		$v_2(b) = 1$	$(2, ?, ?, 1, 1, 2)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \geq 4$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a^2 - \frac{b^2}{16}) \leq 3$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 0$	$(3, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \leq 2$	$(4, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$, $v_2(a + \frac{b}{4}) = 1$, and $v_2(a - \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) = 2$	$(5, ?, ?, ?, 2, 2)$
		$v_2(b) = 2$ and $v_2(a + \frac{b}{4}) \geq 3$	$(5, ?, ?, ?, 3, 3)$
		$v_2(b) \geq 3$	$(3, ?, ?, 2, 3, 2)$

So for the 5th curve of the 8-Isogeny, $v_2(u) = 2$, so $u = 4$ when $a = 117$ and $b = 68$.

Torsion Method

The torsion method works when our elliptic curves $F_{n,i}$ have a non-trivial point of finite order. If this is the case, then there is a classification for the minimal discriminant of such elliptic curves. Consequently, our second method deduces the minimal discriminant of $F_{n,i}$ by using this classification.

Torsion Method Techniques: 6th Isogeny, 2nd Curve

Let $A = 9a$, $B = -9a - b$, and $d = \gcd(A, B)$. Then,

$$F_{6,2} = E_{C_6}(A, B) : y^2 + (a-b)xy - (A^2B + AB^2)y = x^3 - (AB + B^2)x^2$$

By the classification of minimal discriminants of elliptic curves with non-trivial torsion, the minimal discriminant of $F_{6,2}$ is

$$u^{-12}d^{-12}\Delta_{F_{6,2}} \text{ where } u = \begin{cases} 2 & \text{if } \nu_2(\frac{A}{d} + \frac{B}{d}) \geq 3 \\ 1 & \text{if } \nu_2(\frac{A}{d} + \frac{B}{d}) \leq 2 \end{cases}$$

Using the Torison Method: The 6-Isogeny

$$\begin{tikzcd}
F_{6,1} \arrow[r, dash, "2"] \arrow[d, dash, "3"] & F_{6,2} \arrow[d, dash, "3"] \\
F_{6,3} \arrow[r, dash, "2"] & F_{6,4}
\end{tikzcd}$$

Thank you!

Questions?

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Example: 2nd Curve of the 6-Isogeny

$A : 9a \qquad B : -9a - b$

Step 1: If $p \mid \gcd(A, B)$, $p \neq 3$, then

$9a \equiv 0 \mod p \to p \mid a$

$9a - b \equiv 0 \mod p \to p \mid b$

This is a contradiction as a and b are relatively prime.

$3 \mid \gcd(A, B) \to 3 \mid 9a + b \to 3 \mid b$

Example: 2nd Curve of the 6-Isogeny

Step 2:

$$v_3(\gcd(A,B)) = \begin{cases} 0 & \text{if } v_3(b) = 0 \\ 1 & \text{if } v_3(b) = 1 \\ 2 & \text{if } v_3(b) \geq 2 \end{cases}$$

Example: 2nd Curve of the 6-Isogeny

Step 3: Find u' values using Theorem 4.4: $T = C_6$, which has:
$u' = 2$ if $v_2(A + B) \geq 3$
$u' = 1$ if $v_2(A + B) \leq 2$
Note that $A + B = -b$, so $v_2(A + B) = v_2(b)$

Example: 2nd Curve of the 6-Isogeny

Results:

$v_3(b) = 0$ and $v_2(b) \leq 2$, then $u = 1$

$v_3(b) = 0$ and $v_2(b) \geq 3$, then $u = 2$

$v_3(b) = 1$ and $v_2(b) \leq 2$, then $u = 3$

$v_3(b) = 1$ and $v_2(b) \geq 3$, then $u = 6$

$v_3(b) \geq 2$ and $v_2(b) \leq 2$, then $u = 9$

$v_3(b) \geq 2$ and $v_2(b) \geq 3$, then $u = 18$